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Generalized integral operators and Schwartz kernel type theorems

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Received 14 October 2004

Available online 23 February 2005

Submitted by R. Curto

Abstract

In analogy to the classical Schwartz kernel theorem, we show that a large class of linear mappings admits integral kernels in the framework of Colombeau generalized functions. To do this, we introduce new spaces of generalized functions with slow growth and the corresponding adapted linear mappings. Finally, we show that, in some sense, Schwartz' result is contained in our main theorem.

Keywords: Schwartz kernel theorem; Colombeau nonlinear generalized functions; Integral operator; Distributions

1. Introduction

It is well known that the framework of Schwartz distributions is not suitable for posing and solving many differential or integral problems with singular coefficients or data. A natural approach to overcome this difficulty consists in replacing the given problem by a one-parameter family of smooth problems. This is done in most theories of generalized functions and, for example, in Colombeau simplified theory which we are going to use in the sequel. (For details, see the monographs [2,7,12] and the references therein.)

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.01.006

In this paper, we continue the investigations in the field of generalized integral operators initiated by the pioneering work of D. Scarpalezos [15], and carried on by J.-F. Colombeau (personal communications and [1]) in view of applications to physics and by C. Garetto et al. [5,6] with applications to pseudo-differential operators theory and questions of regularity.

More precisely, the following results holds: every H belonging to $\mathcal{G}(\mathbb{R}^m \times \mathbb{R}^n)$ defines a linear operator from $\mathcal{G}_C(\mathbb{R}^n)$ to $\mathcal{G}(\mathbb{R}^m)$ by the formula

$$\tilde{H} : \mathcal{G}_C\big(\mathbb{R}^n\big) \to \mathcal{G}\big(\mathbb{R}^m\big), \quad f \mapsto \tilde{H}(f) = \left[\left(x \longmapsto \int H_\varepsilon(x, y) f_\varepsilon(y)\, dy\right)_\varepsilon\right],$$

where $(H_\varepsilon)_\varepsilon$ (respectively $(f_\varepsilon)_\varepsilon$) is any representative of H (respectively f) and $[\,\cdot\,]$ is the class of an element in $\mathcal{G}(\mathbb{R}^d)$. (For any $d \in \mathbb{N}$, $\mathcal{G}(\mathbb{R}^d)$ denotes the usual quotient space of Colombeau simplified generalized functions, while $\mathcal{G}_C(\mathbb{R}^d)$ is the subspace of elements of $\mathcal{G}(\mathbb{R}^d)$ compactly supported: see Section 2 for the mathematical framework.)

Conversely, in the distributional case, the well-known Schwartz kernel theorem asserts that each linear map Λ from $\mathcal{D}(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^m)$ continuous for the strong topology of $\mathcal{D}'$ can be represented by a kernel $K \in \mathcal{D}'(\mathbb{R}^m \times \mathbb{R}^n)$, that is

$$\forall f \in \mathcal{D}\big(\mathbb{R}^n\big),\ \forall \varphi \in \mathcal{D}\big(\mathbb{R}^m\big), \quad \big(\Lambda(f), \varphi\big) = (K, \varphi \otimes f).$$

Let us recall here that $\mathcal{D}(\mathbb{R}^n)$ is embedded in $\mathcal{G}_C(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^m)$ in $\mathcal{G}(\mathbb{R}^m)$. In the spirit of Schwartz' theorem, we prove that, in the framework of Colombeau generalized functions, any net of linear maps $(L_\varepsilon : \mathcal{D}(\mathbb{R}^n) \to \mathrm{C}^\infty(\mathbb{R}^m))_\varepsilon$, satisfying some growth properties with respect to the parameter ε, gives rise to a linear map $L : \mathcal{G}_C(\mathbb{R}^n) \to \mathcal{G}(\mathbb{R}^m)$ which can be represented as an integral operator. This means that there exists a generalized function $H_L \in \mathcal{G}(\mathbb{R}^m \times \mathbb{R}^n)$ such that, for any f belonging to convenient subspaces of $\mathcal{G}_C(\mathbb{R}^n)$ depending on the regularity of the map $(L_\varepsilon)_\varepsilon$ with respect to ε, we have

$$L(f) = \left[\left(x \longmapsto \int H_{L,\varepsilon}(x, y) f_\varepsilon(y)\, dy\right)_\varepsilon\right],$$

where $(H_{L,\varepsilon})_\varepsilon$ (respectively $(f_\varepsilon)_\varepsilon$) is any representative of H (respectively f).

Moreover, this result is strongly related to the Schwartz kernel theorem in the following sense. We can associate to each linear operator $\Lambda : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^m)$, satisfying the above mentioned hypothesis, a strongly moderate map L_Λ and consequently a kernel $H_{L_\Lambda} \in \mathcal{G}(\mathbb{R}^m \times \mathbb{R}^n)$ with the following equality property: for all f in $\mathcal{D}(\mathbb{R}^n)$, $\Lambda(f)$ and $\tilde{H}_{L_\Lambda}(f)$ are equal in the generalized distribution sense [11], that is, for all $k \in \mathbb{N}$ and $(H_{L_{\Lambda,\varepsilon}})_\varepsilon$ representative of H_{L_Λ},

$$\forall \Phi \in \mathcal{D}\big(\mathbb{R}^m\big), \quad \big\langle \Lambda(f), \Phi \big\rangle - \int \left(\int H_{L_{\Lambda,\varepsilon}}(x, y) f(y)\, dy\right) \Phi(x)\, dx = \mathrm{O}\big(\varepsilon^k\big),$$
$$\text{for } \varepsilon \to 0.$$

The paper can be divided in two parts. The first part, formed by Sections 2 and 3, introduces all the material which is needed in the sequel. We mention here in particular the notion of *spaces of generalized functions with slow growth*, which are subspaces of the usual space $\mathcal{G}(\mathbb{R}^d)$ with additional limited growth property with respect to the parameter ε. Lemma 17 shows one feature of those spaces (used for the proof of the main results):

convolution admits some special δ-nets as unit on them, whereas this result is false in $\mathcal{G}(\mathbb{R}^d)$. The second part, consisting in the two last sections, is devoted to the definition of strongly moderate nets, the statement of the main results and their proofs.

2. Colombeau type algebras

2.1. The sheaf of Colombeau simplified algebras

Let C^∞ be the sheaf of complex valued smooth functions on $\mathbb{R}^d$ ($d \in \mathbb{N}$) with the usual topology of uniform convergence. For every open set Ω of $\mathbb{R}^d$, this topology can be described by the family of seminorms $(p_{K,l}(f))_{K \Subset \Omega,\, l \in \mathbb{N}}$, with

$$p_{K,l}(f) = \sup_{x \in K,\, |\alpha| \leqslant l} \left|\partial^\alpha f(x)\right|.$$

(The notation $K \Subset \Omega$ means that the set K is a compact set included in Ω.)

Set

$$\begin{aligned}
\mathcal{X}\big(\mathrm{C}^\infty(\Omega)\big) = \big\{(f_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\Omega)^{(0,1]} \;\big|\; &\forall K \Subset \Omega,\ \forall l \in \mathbb{N},\ \exists q \in \mathbb{N},\\
&p_{K,l}(f_\varepsilon) = \mathrm{O}\big(\varepsilon^{-q}\big) \text{ for } \varepsilon \to 0\big\},\\
\mathcal{N}\big(\mathrm{C}^\infty(\Omega)\big) = \big\{(f_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\Omega)^{(0,1]} \;\big|\; &\forall K \Subset \Omega,\ \forall l \in \mathbb{N},\ \forall p \in \mathbb{N},\\
&p_{K,l}(f_\varepsilon) = \mathrm{O}\big(\varepsilon^{p}\big) \text{ for } \varepsilon \to 0\big\}.
\end{aligned}$$

Lemma 1 [9,10].

(i) *The functor* $\mathcal{X} : \Omega \to \mathcal{X}(\mathrm{C}^\infty(\Omega))$ *defines a sheaf of subalgebras of the sheaf* $(\mathrm{C}^\infty)^{(0,1]}$.
(ii) *The functor* $\mathcal{N} : \Omega \to \mathcal{N}(\mathrm{C}^\infty(\Omega))$ *defines a sheaf of ideals of the sheaf* $\mathcal{X}$.

The proof of this lemma is mainly based on the two following arguments:

(a) For each open subset Ω of X, we have

$$\begin{aligned}
&\forall l \in \mathbb{N},\ \forall K \Subset \Omega,\ \exists C \in \mathbb{R}_+^*,\ \forall (f,g) \in \big(\mathrm{C}^\infty(\Omega)\big)^2,\\
&\quad p_{K,l}(fg) \leqslant C p_{K,l}(f) p_{K,l}(g),
\end{aligned}$$

which asserts that the $(p_{K,l})_{K \Subset \Omega,\, l \in \mathbb{N}}$-topology of $\mathrm{C}^\infty(\Omega)$ is compatible with its algebraic structure.

(b) For two open subsets $\Omega_1 \subset \Omega_2$ of $\mathbb{R}^d$, the family of seminorms $(p_{K,l})$ related to Ω_1 is included in the family of seminorms related to Ω_2, and

$$\forall l \in \mathbb{N},\ \forall K \Subset \Omega_1,\ \forall f \in \mathrm{C}^\infty(\Omega_2),\quad p_{K,l}(f_{|\Omega_1}) = p_{K,l}(f).$$

Definition 2. The sheaf of factor algebras

$$\mathcal{G} = \mathcal{X}\big(\mathrm{C}^\infty(\cdot)\big) / \mathcal{N}\big(\mathrm{C}^\infty(\cdot)\big)$$

is called the sheaf of *Colombeau type algebras*.

The sheaf $\mathcal{G}$ turns out to be a sheaf of differential algebras and a sheaf of modules on the factor ring $\bar{\mathbb{C}} = \mathcal{X}(\mathbb{C})/\mathcal{N}(\mathbb{C})$, with

$$\mathcal{X}(\mathbb{K}) = \left\{(r_\varepsilon)_\varepsilon \in \mathbb{K}^{(0,1]} \;\middle|\; \exists q \in \mathbb{N},\ |r_\varepsilon| = \mathrm{O}\big(\varepsilon^{-q}\big) \text{ for } \varepsilon \to 0\right\},$$
$$\mathcal{N}(\mathbb{K}) = \left\{(r_\varepsilon)_\varepsilon \in \mathbb{K}^{(0,1]} \;\middle|\; \forall p \in \mathbb{N},\ |r_\varepsilon| = \mathrm{O}\big(\varepsilon^{p}\big) \text{ for } \varepsilon \to 0\right\},$$

where $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R},\ \mathbb{R}_+$.

Notation 3. In the sequel, we shall note, as usual, $\mathcal{G}(\Omega)$ instead of $\mathcal{G}(\mathrm{C}^\infty(\Omega))$ the algebra of generalized functions on Ω. For $(f_\varepsilon)_\varepsilon \in \mathcal{X}(\mathrm{C}^\infty(\Omega))$, $[(f_\varepsilon)_\varepsilon]$ will be its class in $\mathcal{G}(\Omega)$.

2.2. Generalized functions with compact supports

Let us mention here some remarks about generalized functions with compact supports, which will be useful in the sequel.

As $\mathcal{G}$ is a sheaf, the notion of support of a section $f \in \mathcal{G}(\Omega)$ (Ω open subset of $\mathbb{R}^d$) makes sense. The following definition will be sufficient for this paper.

Definition 4. The support of a generalized function $f \in \mathcal{G}(\Omega)$, denoted by $\operatorname{supp} f$, is the complement in Ω of the largest open subset of Ω on which f is null.

Notation 5. We denote by $\mathcal{G}_C(\Omega)$ the subset of $\mathcal{G}(\Omega)$ of elements with compact support.

Lemma 6. *Every $f \in \mathcal{G}_C$ has a representative $(f_\varepsilon)_\varepsilon$ such that each f_ε has the same compact support.*

For our subject, there is a more convenient way to introduce generalized functions with compact support. We start from the algebra $\mathcal{D}(\Omega)$ considered as the inductive limit of

$$\mathcal{D}_j(\Omega) = \mathcal{D}_{K_j}(\Omega) = \left\{f \in \mathcal{D}(\Omega) \;\middle|\; \operatorname{supp} f \subset K_j\right\},$$

where:

(i) $(K_j)_{j\in\mathbb{N}}$ is an increasing sequence of relatively compact subsets exhausting Ω, with $K_j \subset \mathring{K}_{j+1}$;
(ii) $\mathcal{D}_j(\Omega)$ is endowed with the family of seminorms $(p_{j,l})_{l\in\mathbb{N}}$ defined by

$$p_{j,l}(f) = \sup_{x\in K_j,\, |\alpha|\leqslant l} \left|\partial^\alpha f(x)\right|.$$

The topology on $\mathcal{D}(\Omega)$ does not depend on the particular choice of the sequence $(K_j)_{j\in\mathbb{N}}$. Construction of spaces of generalized functions based on projective or inductive limits have already been considered (see, e.g., [3,14]). We just recall it briefly here. Let $(K_j)_{j\in\mathbb{N}}$ be a fixed sequence of compact sets satisfying (i) and set

$$\mathcal{X}\big(\mathcal{D}(\Omega)\big) = \bigcup_{j\in\mathbb{N}} \mathcal{X}_j(\Omega)$$

$$\text{with } \mathcal{X}_j(\Omega) = \big\{(f_\varepsilon)_\varepsilon \in \mathcal{D}_j(\Omega)^{(0,1]} \,\big|\, \forall l \in \mathbb{N},\ \exists q \in \mathbb{N},\ p_{j,l}(f_\varepsilon) = \mathrm{O}\big(\varepsilon^{-q}\big) \text{ for } \varepsilon \to 0\big\},$$

$$\mathcal{N}\big(\mathcal{D}(\Omega)\big) = \bigcup_{j\in\mathbb{N}} \mathcal{N}_j(\Omega)$$

$$\text{with } \mathcal{N}_j(\Omega) = \big\{(f_\varepsilon)_\varepsilon \in \mathcal{D}_j(\Omega)^{(0,1]} \,\big|\, \forall l \in \mathbb{N},\ \forall p \in \mathbb{N},\ p_{j,l}(f_\varepsilon) = \mathrm{O}\big(\varepsilon^{p}\big) \text{ for } \varepsilon \to 0\big\}. \tag{1}$$

With these definitions, we have:

Lemma 7. *$\mathcal{X}(\mathcal{D}(\Omega))$ is a subalgebra of $\mathcal{D}(\Omega)^{(0,1]}$ and $\mathcal{N}(\mathcal{D}(\Omega))$ an ideal of $\mathcal{X}(\mathcal{D}(\Omega))$.*

The factor space $\mathcal{G}_{\mathcal{D}}(\Omega) = \mathcal{X}(\mathcal{D}(\Omega))/\mathcal{N}(\mathcal{D}(\Omega))$ appears to be a natural space of generalized functions with compact support. The algebra $\mathcal{G}_{\mathcal{D}}(\Omega)$ does not depend on the particular choice of the sequence $(K_j)_{j\in\mathbb{N}}$. Moreover, due to the properties of the family $(p_{j,l})$, we have:

Lemma 8. *The spaces $\mathcal{G}_{\mathcal{D}}(\Omega)$ and $\mathcal{G}_C(\Omega)$ are isomorphic.*

Proof. We use the following fundamental property: for all $j \in \mathbb{N}$ and all $(f_\varepsilon)_\varepsilon \in \mathcal{X}_j(\Omega)$ we have

$$\forall l \in \mathbb{N},\ \forall j' \leqslant j,\ \forall j'' \geqslant j, \quad p_{j',l}(f_\varepsilon) \leqslant p_{j,l}(f_\varepsilon) = p_{j'',l}(f_\varepsilon). \tag{2}$$

The last equality holds since $\operatorname{supp} f \subset K_j \subset K_{j''}$, for all $j'' \geqslant j$.

Relation (2) implies that $\mathcal{X}(\mathcal{D}(\Omega)) \subset \mathcal{X}(\mathrm{C}^\infty(\Omega))$ and $\mathcal{N}(\mathcal{D}(\Omega)) \subset \mathcal{N}(\mathrm{C}^\infty(\Omega))$. Let us show the first inclusion. Consider $(f_\varepsilon)_\varepsilon$ in some $\mathcal{X}_j(\Omega)$. Then, for all $l \in \mathbb{N}$, there exists $q \in \mathbb{N}$ such that: $p_{j,l}(f_\varepsilon) = \mathrm{O}(\varepsilon^{-q})$ for $\varepsilon \to 0$. It follows that $\forall K \Subset \Omega$, $p_{K,l}(f_\varepsilon) \leqslant p_{j,l}(f_\varepsilon) = \mathrm{O}(\varepsilon^{-q})$.

These two inclusions imply that the map

$$\iota : \mathcal{G}_{\mathcal{D}}(\Omega) \to \mathcal{G}(\Omega), \quad (f_\varepsilon)_\varepsilon + \mathcal{N}\big(\mathcal{D}(\Omega)\big) \mapsto (f_\varepsilon)_\varepsilon + \mathcal{N}\big(\mathrm{C}^\infty(\Omega)\big)$$

is well defined, with $\iota(\mathcal{G}_{\mathcal{D}}(\Omega)) \subset \mathcal{G}_C(\Omega)$.

It remains to show that the map ι is bijective. Indeed, if $(f_\varepsilon)_\varepsilon \in \mathcal{N}(\mathrm{C}^\infty(\Omega))$ with $(f_\varepsilon)_\varepsilon \in \mathcal{X}_j(\Omega)$, we have $(f_\varepsilon)_\varepsilon \in \mathcal{N}_j(\Omega)$ and $(f_\varepsilon)_\varepsilon \in \mathcal{N}(\mathcal{D}(\Omega))$. Injectivity follows. Conversely, take $g \in \mathcal{G}_C(\Omega)$. According to Lemma 6, there exists a compact set K and a representative $(g_\varepsilon)_\varepsilon$ of g such that $\operatorname{supp} g_\varepsilon \subset K$, for all ε. We observe that K is included in some K_j, and then, that $(g_\varepsilon)_\varepsilon \in \mathcal{X}_j(\Omega)$. Finally, $\iota((g_\varepsilon)_\varepsilon + \mathcal{N}(\mathcal{D}(\Omega))) = g$. $\square$

2.3. Embeddings

The space $\mathrm{C}^\infty(\mathbb{R}^d)$ ($d \in \mathbb{N}$) is embedded in $\mathcal{G}(\mathbb{R}^d)$ by the canonical map

$$\sigma : \mathrm{C}^\infty\big(\mathbb{R}^d\big) \to \mathcal{G}\big(\mathbb{R}^d\big), \quad f \mapsto (f_\varepsilon)_\varepsilon + \mathcal{N}\big(\mathrm{C}^\infty\big(\mathbb{R}^d\big)\big),$$
$$\text{with } f_\varepsilon = f \text{ for all } \varepsilon \in (0,1],$$

which is an injective homomorphism of algebras.

Moreover, the construction of $\mathcal{G}(\mathbb{R}^d)$ permits to embed the space $\mathcal{D}'(\mathbb{R}^d)$ by means of convolution with suitable mollifiers. We follow in this paper the ideas of [11].

Lemma 9. *There exists a net of mollifiers $(\theta_\varepsilon)_\varepsilon \in \mathcal{D}(\mathbb{R}^d)^{(0,1]}$ such that for all $k \in \mathbb{N}$,*

$$\int \theta_\varepsilon(x)\,dx = 1 + \mathrm{O}\bigl(\varepsilon^k\bigr) \quad \textit{for } \varepsilon \to 0, \tag{3}$$

$$\forall m \in \mathbb{N}^d \backslash \{0\}, \quad \int x^m \theta_\varepsilon(x)\,dx = \mathrm{O}\bigl(\varepsilon^k\bigr) \quad \textit{for } \varepsilon \to 0. \tag{4}$$

Such a net is built in the following way: consider $\rho \in \mathcal{S}(\mathbb{R}^d)$ such that $\int \rho(x)\,dx = 1$, $\int x^m \rho(x)\,dx = 0$ for all $m \in \mathbb{N}^d \backslash \{0\}$ and $\kappa \in \mathcal{D}(\mathbb{R}^d)$ such that $0 \leqslant \kappa \leqslant 1$, $\kappa = 1$ on $[-1,1]^d$ and $\kappa = 0$ on $\mathbb{R}^d \backslash [-2,2]^d$. Then $(\theta_\varepsilon)_\varepsilon$ defined by

$$\forall \varepsilon \in (0,1],\ \forall x \in \mathbb{R}^d, \quad \theta_\varepsilon(x) = \frac{1}{\varepsilon^d} \rho\left(\frac{x}{\varepsilon}\right) \kappa\bigl(x|\ln \varepsilon|\bigr)$$

satisfies conditions of Lemma 9.

Proposition 10. *With notations of Lemma* 9*, the map*

$$\iota : \mathcal{D}'\bigl(\mathbb{R}^d\bigr) \to \mathcal{G}\bigl(\mathbb{R}^d\bigr), \quad T \mapsto (T * \theta_\varepsilon)_\varepsilon + \mathcal{N}\bigl(\mathrm{C}^\infty\bigl(\mathbb{R}^d\bigr)\bigr)$$

is an injective homomorphism of vector spaces. Moreover, $\iota_{|\mathrm{C}^\infty(\Omega)} = \sigma$.

This proposition asserts that the following diagram is commutative:

$$\begin{tikzcd} \mathrm{C}^\infty\bigl(\mathbb{R}^d\bigr) \arrow[r] \arrow[dr, "\sigma"'] & \mathcal{D}'\bigl(\mathbb{R}^d\bigr) \arrow[d, "\iota"] \\ & \mathcal{G}\bigl(\mathbb{R}^d\bigr). \end{tikzcd}$$

2.4. Generalized integral operators

We collect here results about generalized integral operators. We refer the reader to [1,6] for details.

Definition 11. Let H be in $\mathcal{G}(\mathbb{R}^m \times \mathbb{R}^n)$. The integral operator of kernel H is the map $\tilde{H}$ defined by

$$\tilde{H} : \mathcal{G}_C\bigl(\mathbb{R}^n\bigr) \to \mathcal{G}\bigl(\mathbb{R}^m\bigr), \quad f \mapsto \tilde{H}(f) = \left[\left(x \mapsto \int_W H_\varepsilon(x,y) f_\varepsilon(y)\,dy\right)_\varepsilon\right],$$

where $(H_\varepsilon)_\varepsilon$ (respectively $(f_\varepsilon)_\varepsilon$) is any representative of H (respectively f) and W any relatively compact open neighborhood of $\operatorname{supp} f$.

Note that in the above mentioned references, the generalized function H satisfies some additional conditions such as being properly supported. This assumption is not needed in this paper, since we consider operators on $\mathcal{G}_C(\mathbb{R}^n)$: the integral in Definition 11 is performed on a relatively compact set and $\tilde{H}(f)$ does not depend on the choice of such a set.

Proposition 12. *With the notations of Definition* 11, *the operator* $\tilde{H}$ *defines a linear mapping from* $\mathcal{G}_C(\mathbb{R}^n)$ *to* $\mathcal{G}(\mathbb{R}^m)$, *continuous for the respective sharp topologies of* $\mathcal{G}_C(\mathbb{R}^n)$ *and* $\mathcal{G}(\mathbb{R}^m)$.

Moreover, the map

$$\mathcal{G}(\mathbb{R}^m \times \mathbb{R}^n) \to \mathcal{L}(\mathcal{G}_C(\mathbb{R}^n), \mathcal{G}(\mathbb{R}^m)), \quad H \mapsto \tilde{H},$$

is injective.

In other words, the map $\tilde{H}$ is characterized by the kernel H:

$$\tilde{H} = 0 \quad \text{in } \mathcal{L}(\mathcal{G}_C(\mathbb{R}^n), \mathcal{G}(\mathbb{R}^m)) \quad \Longleftrightarrow \quad H = 0 \quad \text{in } \mathcal{G}(\mathbb{R}^m \times \mathbb{R}^n).$$

3. Spaces of generalized functions with slow growth

In the sequel, we need to consider some subspaces of $\mathcal{G}(\Omega)$ (Ω open subset of $\mathbb{R}^d$) with restrictive conditions of growth with respect to $1/\varepsilon$ when the l index of the families of seminorms is involved, that is the index related to derivatives. We show that these spaces give a good framework for extension of linear maps and for convolution of generalized functions. These are essential properties for our result.

3.1. Definitions

Set

$$\begin{aligned}\mathcal{X}_{\mathcal{L}_0}(\mathrm{C}^\infty(\Omega)) = \Big\{(f_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\Omega)^{(0,1]} \;\Big|\; & \forall K \Subset \Omega,\ \exists q \in \mathbb{N}^{\mathbb{N}}, \\ & \text{with } \lim_{l\to+\infty} (q(l)/l) = 0, \\ & \forall l \in \mathbb{N},\ p_{K,l}(f_\varepsilon) = \mathrm{O}(\varepsilon^{-q(l)}) \text{ for } \varepsilon \to 0\Big\},\end{aligned}$$

and, for $a \in (0, +\infty]$,

$$\begin{aligned}\mathcal{X}_{\mathcal{L}_a}(\mathrm{C}^\infty(\Omega)) = \Big\{(f_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\Omega)^{(0,1]} \;\Big|\; & \forall K \Subset \Omega,\ \exists q \in \mathbb{N}^{\mathbb{N}}, \\ & \text{with } \lim\sup_{l\to+\infty} (q(l)/l) < a, \\ & \forall l \in \mathbb{N},\ p_{K,l}(f_\varepsilon) = \mathrm{O}(\varepsilon^{-q(l)}) \text{ for } \varepsilon \to 0\Big\}.\end{aligned} \tag{5}$$

Lemma 13. *For all* $a \in (0, +\infty]$, *a net* $(f_\varepsilon)_\varepsilon \in \mathcal{X}(\Omega)^{(0,1]}$ *belongs to* $\mathcal{X}_{\mathcal{L}_a}(\mathrm{C}^\infty(\Omega))$ *iff for all* $K \Subset \Omega$, *there exists* $(a', b) \in (\mathbb{R}_+)^2$ *with* $a' < a$ *such that*

$$\forall l \in \mathbb{N}, \quad p_{K,l}(f_\varepsilon) = \mathrm{O}(\varepsilon^{-a'l-b}) \quad \textit{for } \varepsilon \to 0.$$

The proof is straightforward and left to the reader. In other words, this lemma asserts that the growth of the sequence $q(l)$ which appears in (5) is at most linear.

Lemma 14. *For all $a \in [0, +\infty]$, $\mathcal{X}_{\mathcal{L}_a}(\mathrm{C}^\infty(\Omega))$ is a subalgebra of $\mathcal{X}(\mathrm{C}^\infty(\Omega))$ over the ring $\mathcal{X}(\mathbb{C})$.*

Proof. We shall do the proof for $a \in (0, +\infty]$. Take $(f_\varepsilon)_\varepsilon$ and $(g_\varepsilon)_\varepsilon$ in $\mathcal{X}_{\mathcal{L}_a}(\mathrm{C}^\infty(\Omega))$ and $K \Subset \Omega$. According to Lemma 13, there exists $(a', b) \in (\mathbb{R}_+)^2$ with $a' < a$ such that

$$\forall l \in \mathbb{N}, \quad p_{K,l}(h_\varepsilon) = \mathrm{O}\big(\varepsilon^{-a'l-b}\big) \quad \text{for } \varepsilon \to 0 \text{ for } h_\varepsilon = f_\varepsilon \text{ and } h_\varepsilon = g_\varepsilon.$$

We get immediately that $p_{K,l}(f_\varepsilon + g_\varepsilon) = \mathrm{O}(\varepsilon^{-a'l-b})$ for $\varepsilon \to 0$, and that $(f_\varepsilon + g_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{L}_a}(\mathrm{C}^\infty(\Omega))$.

For $(c_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{C})$, there exists q_c such that $|c_\varepsilon| = \mathrm{O}(\varepsilon^{-q_c})$ for $\varepsilon \to 0$. Then $p_{K,l}(c_\varepsilon f_\varepsilon) = \mathrm{O}(\varepsilon^{-a'l-b-q_c})$ and $(c_\varepsilon f_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{L}_a}(\mathrm{C}^\infty(\Omega))$. It follows that $\mathcal{X}_{\mathcal{L}_a}(\mathrm{C}^\infty(\Omega))$ is a submodule of $\mathcal{X}(\mathrm{C}^\infty(\Omega))$ over $\mathcal{X}(\mathbb{C})$.

Consider now $l \in \mathbb{N}$ and $\alpha \in \mathbb{N}^d$ with $|\alpha| = l$. By the Leibniz formula, we have

$$\forall \varepsilon \in (0, 1], \quad \partial^\alpha(f_\varepsilon g_\varepsilon) = \sum_{\gamma \leqslant \alpha} C_\alpha^\gamma \partial^\gamma f_\varepsilon \partial^{\alpha-\gamma} g_\varepsilon,$$

where C_α^γ is the generalized binomial coefficient. Thus

$$\sup_{x \in K} \big|\partial^\alpha(f_\varepsilon g_\varepsilon)(x)\big| \leqslant \sum_{\gamma \leqslant \alpha} C_\alpha^\gamma \, p_{K,|\gamma|}(f_\varepsilon) p_{K,|\alpha-\gamma|}(f_\varepsilon) = \mathrm{O}\big(\varepsilon^{-a'(|\gamma|+|\alpha-\gamma|)-2b}\big) \quad \text{for } \varepsilon \to 0.$$

As $\gamma \leqslant \alpha$, we get $|\gamma| + |\alpha - \gamma| = |\alpha| = l$ and

$$\sup_{x \in K} \big|\partial^\alpha(f_\varepsilon g_\varepsilon)(x)\big| = \mathrm{O}\big(\varepsilon^{-a'|\alpha|-2b}\big) \quad \text{for } \varepsilon \to 0.$$

Thus, $p_{K,l}(f_\varepsilon\, g_\varepsilon) = \mathrm{O}(\varepsilon^{-a'|\alpha|-2b})$ for $\varepsilon \to 0$, and $(f_\varepsilon\, g_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{L}_a}(\mathrm{C}^\infty(\Omega))$. $\square$

Consequently, we can consider the subalgebras of $\mathcal{G}(\Omega)$ defined by

$$\mathcal{G}_{\mathcal{L}_a}(\Omega) = \mathcal{X}_{\mathcal{L}_a}\big(\mathrm{C}^\infty(\Omega)\big)/\mathcal{N}\big(\mathrm{C}^\infty(\Omega)\big).$$

Remark 15.

(i) For $a < b$, we have $\mathcal{X}_{\mathcal{L}_a} \subset \mathcal{X}_{\mathcal{L}_b}$ and thus $\mathcal{G}_{\mathcal{L}_a}(\Omega) \subset \mathcal{G}_{\mathcal{L}_b}(\Omega)$.

(ii) Some spaces with more restrictive conditions of growth with respect to the parameter ε have already been considered (see, e.g., [12,15]). Set

$$\mathcal{X}^\infty\big(\mathrm{C}^\infty(\Omega)\big) = \big\{(f_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\Omega)^{(0,1]} \;\big|\; \forall K \Subset \Omega,\ \exists q \in \mathbb{N},\ \forall l \in \mathbb{N},\ p_{K,l}(f_\varepsilon) = \mathrm{O}\big(\varepsilon^{-q}\big) \text{ for } \varepsilon \to 0\big\}.$$

$\mathcal{X}^\infty(\mathrm{C}^\infty(\Omega))$ turns out to be a subalgebra of $\mathcal{X}_{\mathcal{L}_a}(\mathrm{C}^\infty(\Omega))$, for all $a \in [0, +\infty]$. Therefore

$$\mathcal{G}^\infty(\Omega) = \mathcal{X}^\infty\big(\mathrm{C}^\infty(\Omega)\big)/\mathcal{N}\big(\mathrm{C}^\infty(\Omega)\big)$$

is a subalgebra of $\mathcal{G}_{\mathcal{L}_a}(\Omega)$ and $\mathcal{G}(\Omega)$. For the local analysis or microlocal analysis of generalized functions, $\mathcal{G}^\infty$ plays the role of C^∞ in the case of distributions [11,13]. Our spaces $\mathcal{G}_{\mathcal{L}_a}(\Omega)$ give new types of regularity for generalized functions. This will be studied in a forthcoming paper.

Notation 16. We shall note $\mathcal{G}_C^\infty(\Omega)$ (respectively $\mathcal{G}_{\mathcal{L}_a,C}(\Omega)$) the subspace of compactly supported elements of $\mathcal{G}^\infty(\Omega)$ (respectively $\mathcal{G}_{\mathcal{L}_a}(\Omega)$).

3.2. Fundamental lemma

Lemma 17. *Let a be a real in $[0,1]$, d be a positive integer and $(\theta_\varepsilon)_\varepsilon \in \mathcal{D}(\mathbb{R}^d)^{(0,1]}$ a net of mollifiers satisfying conditions* (3) *and* (4). *For any $(g_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{L}_a}(\mathrm{C}^\infty(\mathbb{R}^d))$, we have*

$$(g_\varepsilon * \theta_\varepsilon - g_\varepsilon)_\varepsilon \in \mathcal{N}\big(C^\infty\big(\mathbb{R}^d\big)\big). \tag{6}$$

Proof. We shall prove this lemma in the case $d=1$, the general case only differs by more complicated algebraic expressions. It suffices to treat the case $a=1$, since $\mathcal{X}_{\mathcal{L}_a} \subset \mathcal{X}_{\mathcal{L}_1}$, as mentioned in Remark 15.

Fix $(g_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{L}_1}(\mathrm{C}^\infty(\mathbb{R}^d))$, K a compact set of $\mathbb{R}$ and set $\Delta_\varepsilon = g_\varepsilon * \theta_\varepsilon - g_\varepsilon$ for $\varepsilon \in (0,1]$. Writing $\int \theta_\varepsilon(x)\,dx = 1 + \mathcal{N}_\varepsilon$ with $(\mathcal{N}_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R})$, we get

$$\begin{aligned}\Delta_\varepsilon(y) &= \int g_\varepsilon(y-x)\theta_\varepsilon(x)\,dx - g_\varepsilon(y)\\ &= \int \big(g_\varepsilon(y-x) - g_\varepsilon(y)\big)\theta_\varepsilon(x)\,dx + \mathcal{N}_\varepsilon g_\varepsilon(y).\end{aligned}$$

The integration is performed on the compact set $[-2/|\ln\varepsilon|, 2/|\ln\varepsilon|]$ which contains $\operatorname{supp}\theta_\varepsilon$.

There exists a compact set K' with $[y-1, y+1] \subset K'$ for all $y \in K$, and a sequence $q: \mathbb{N} \to \mathbb{N}$ with $\limsup_{i\to+\infty}(q(i)/i) < 1$ such that

$$\forall i \in \mathbb{N}, \quad \sup_{\xi \in K'} \big|g_\varepsilon^{(i)}(\xi)\big| = \mathrm{O}\big(\varepsilon^{-q(i)}\big) \quad \text{for } \varepsilon \to 0.$$

Let m be a positive integer. As $\limsup_{i\to+\infty}(q(i)/i) < 1$, we get $\lim_{i\to+\infty}(i - q(i)) = +\infty$ and the existence of an integer k such that $k - q(k) > m$. Taylor's formula gives

$$g_\varepsilon(y-x) - g_\varepsilon(y) = \sum_{i=1}^{k-1} \frac{(-x)^i}{i!} g_\varepsilon^{(i)}(y) + \frac{(-x)^k}{(k-1)!} \int_0^1 g_\varepsilon^{(k)}(y-ux)(1-u)^{k-1}\,du,$$

and

$$\Delta_\varepsilon(y) = \underbrace{\sum_{i=1}^{k-1} \frac{(-1)^i}{i!} g_\varepsilon^{(i)}(y) \int x^i \theta_\varepsilon(x)\,dx}_{P_\varepsilon(k,y)}$$
$$+ \underbrace{\int_{-2/|\ln\varepsilon|}^{2/|\ln\varepsilon|} \frac{(-x)^k}{(k-1)!} \int_0^1 g_\varepsilon^{(k)}(y-ux)(1-u)^{k-1}\,du\,\theta_\varepsilon(x)\,dx}_{R_\varepsilon(k,y)} + \mathcal{N}_\varepsilon g_\varepsilon^{(k)}(y).$$

According to Lemma 9, we have $(\int x^i \theta_\varepsilon(x)\,dx)_\varepsilon \in \mathcal{N}(\mathbb{R})$ and consequently

$$\forall i \in \{0,\dots,k-1\}, \quad \int x^i \theta_\varepsilon(x)\,dx = \mathrm{O}\big(\varepsilon^{m+q(i)}\big) \quad \text{for } \varepsilon \to 0.$$

We get

$$\sup_{y\in K} \big|P_\varepsilon(k,y)\big| = \mathrm{O}\big(\varepsilon^m\big) \quad \text{for } \varepsilon \to 0.$$

Using the definition of θ_ε, we have

$$R_\varepsilon(k,y) = \frac{1}{\varepsilon} \int_{-2/|\ln\varepsilon|}^{2/|\ln\varepsilon|} \frac{(-x)^k}{(k-1)!} \left(\int_0^1 g_\varepsilon^{(k)}(y-ux)(1-u)^{k-1}\,du \right)$$
$$\times \rho\left(\frac{x}{\varepsilon}\right) \chi\big(x|\ln\varepsilon|\big)\,dx.$$

Setting $v = x/\varepsilon$, we get

$$R_\varepsilon(k,y) = \frac{\varepsilon^k}{(k-1)!} \int_{-2/(\varepsilon|\ln\varepsilon|)}^{2/(\varepsilon|\ln\varepsilon|)} (-v)^k \left(\int_0^1 g_\varepsilon^{(k)}(y-\varepsilon uv)(1-u)^{k-1}\,du \right)$$
$$\times \rho(v) \chi\big(\varepsilon|\ln\varepsilon|v\big)\,dv.$$

For $(u,v) \in [0,1] \times [-2/(\varepsilon|\ln\varepsilon|), 2/(\varepsilon|\ln\varepsilon|)]$, we have $y - \varepsilon uv \in [y-1, y+1]$ for ε small enough. Then, for $y \in K$, $y - \varepsilon uv$ lies in a compact set K' for (u,v) in the domain of integration.

It follows

$$\big|R_\varepsilon(k,y)\big| \leqslant \frac{\varepsilon^k}{(k-1)!} \sup_{\xi\in K'} \big|g_\varepsilon^{(k)}(\xi)\big| \int_{-2/(\varepsilon|\ln\varepsilon|)}^{2/(\varepsilon|\ln\varepsilon|)} |v|^k \big|\rho(v)\big|\,dv$$
$$\leqslant \frac{\varepsilon^k}{(k-1)!} \sup_{\xi\in K'} \big|g_\varepsilon^{(k)}(\xi)\big| \int_{-\infty}^{+\infty} |v|^k \big|\rho(v)\big|\,dv$$
$$\leqslant C \sup_{\xi\in K'} \big|g_\varepsilon^{(k)}(\xi)\big| \varepsilon^k \quad (C > 0).$$

The constant C depends only on the integer k and on ρ. By assumption on k, we get

$$\sup_{y\in K}\left|R_\varepsilon(k,y)\right| = \mathrm{O}\left(\varepsilon^m\right) \quad \text{for } \varepsilon \to 0.$$

Summing up all results, we get $\sup_{y\in K} \Delta_\varepsilon(y) = \mathrm{O}(\varepsilon^m)$ for $\varepsilon \to 0$.

As $(\Delta_\varepsilon)_\varepsilon \in \mathcal{X}(\mathrm{C}^\infty(\mathbb{R}^d))$ and $\sup_{y\in K} \Delta_\varepsilon(y) = \mathrm{O}(\varepsilon^m)$ for $\varepsilon \to 0$, for all $m > 0$ and $K \Subset \mathbb{R}$, we can conclude that $(\Delta_\varepsilon)_\varepsilon \in \mathcal{N}(\mathrm{C}^\infty(\mathbb{R}^d))$, without estimating the derivatives by using [7, Theorem 1.2.3]. □

Remark 18. Let us fix a net of mollifiers $(\theta_\varepsilon)_\varepsilon$, satisfying conditions (3) and (4), to embed $\mathcal{D}'(\mathbb{R}^d)$ in $\mathcal{G}(\mathbb{R}^d)$. Relation (6) shows that $[(\theta_\varepsilon)_\varepsilon]$ plays the role of identity for convolution in $\mathcal{G}_{\mathcal{L}_a}(\mathbb{R}^d)$, whereas this is not true for $\mathcal{G}(\mathbb{R}^d)$. This is an essential feature of these new spaces. (See also Example 25 below.)

4. Schwartz type theorem

4.1. *Extension of linear maps*

Nets of maps $(L_\varepsilon)_\varepsilon$ between two topological algebras having some good growth properties with respect to the parameter ε, can be extended to the respective Colombeau spaces based on algebras, as it is shown in [4,7], for example. We are going to introduce here new notions adapted to our framework.

We use notations of 2.2, especially

$$\mathcal{D}_j\left(\mathbb{R}^n\right) = \left\{ f \in \mathcal{D}\left(\mathbb{R}^n\right) \;\middle|\; \operatorname{supp} f \subset K_j \right\},$$

where $(K_j)_{j\in\mathbb{N}}$ is a sequence of compact sets exhausting $\mathbb{R}^n$, and $\mathcal{D}_j(\mathbb{R}^n)$ is endowed with the family of seminorms $p_{j,l}(f) = \sup_{x\in K_j, |\alpha|\leqslant l} |\partial^\alpha f(x)|$.

Definition 19. Let j be an integer and $(L_\varepsilon)_\varepsilon \in \mathcal{L}(\mathcal{D}_j(\mathbb{R}^n), \mathrm{C}^\infty(\mathbb{R}^m))^{(0,1]}$ be a net of linear maps.

(i) We say that $(L_\varepsilon)_\varepsilon$ is *continuously moderate* (respectively *negligible*) if

$$\begin{aligned}
&\forall K \Subset \mathbb{R}^m,\ \forall l \in \mathbb{N},\ \exists (C_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{R}_+) \text{ (respectively } \mathcal{N}(\mathbb{R}_+)),\\
&\exists l' \in \mathbb{N},\ \forall f \in \mathcal{D}_j\left(\mathbb{R}^n\right),\\
&\quad p_{K,l}(L_\varepsilon(f)) \leqslant C_\varepsilon p_{j,l'}(f), \quad \text{for } \varepsilon \text{ small enough.}
\end{aligned} \tag{7}$$

(ii) Let (b, c) be in $(\mathbb{R}_+ \cup \{+\infty\}) \times \mathbb{R}_+$. We say that $(L_\varepsilon)_\varepsilon$ is $\mathcal{L}_{b,c}$*-strongly continuously moderate* if

$$\begin{aligned}
&\forall K \Subset \mathbb{R}^m,\ \exists \lambda \in \mathbb{N}^{\mathbb{N}} \text{ with } \limsup_{l\to+\infty}(\lambda(l)/l) < b,\\
&\exists r \in \mathbb{N}^{\mathbb{N}} \text{ with } \limsup_{l\to+\infty}(r(l)/l) < c,
\end{aligned}$$

$$\forall l \in \mathbb{N},\ \exists C \in \mathbb{R}_+,\ \forall f \in \mathcal{D}_j\big(\mathbb{R}^n\big), \quad p_{K,l}\big(L_\varepsilon(f)\big) \leqslant C\varepsilon^{-r(l)} p_{j,\lambda(l)}(f),$$
$$\text{for } \varepsilon \text{ small enough.} \tag{8}$$

For the strong moderation, more precise estimates are given for the constants which appear in (7). As our main result is based on linear maps from $\mathcal{D}(\mathbb{R}^n)$ to $\mathrm{C}^\infty(\mathbb{R}^m)$, we need one further extension:

Definition 20. A net of maps $(L_\varepsilon)_\varepsilon \in \mathcal{L}(\mathcal{D}(\mathbb{R}^n), \mathrm{C}^\infty(\mathbb{R}^m))^{(0,1]}$ is *continuously moderate* (respectively *negligible*, $\mathcal{L}_{b,c}$*-strongly continuously moderate*) if for every $j \in \mathbb{N}$, the restriction $(L_{\varepsilon|\mathcal{D}_j(\mathbb{R}^n)}) \in \mathcal{L}(\mathcal{D}_j(\mathbb{R}^n), \mathrm{C}^\infty(\mathbb{R}^m))^{(0,1]}$ is continuously moderate (respectively negligible, $\mathcal{L}_{b,c}$-strongly continuously moderate) in the sense of Definition 19.

Proposition 21.

(i) *Any continuously moderate net* $(L_\varepsilon)_\varepsilon \in (\mathcal{L}(\mathcal{D}(\mathbb{R}^n), \mathrm{C}^\infty(\mathbb{R}^m)))^{(0,1]}$ *can be extended to a map* $L \in \mathcal{L}(\mathcal{G}_C(\mathbb{R}^n), \mathcal{G}(\mathbb{R}^m))$ *defined by*
$$L(f) = \big(L_\varepsilon(f_\varepsilon)\big)_\varepsilon + \mathcal{N}\big(\mathrm{C}^\infty\big(\mathbb{R}^m\big)\big), \tag{9}$$
where $(f_\varepsilon)_\varepsilon$ *is any representative of* f.

(ii) *The extension* L *depends on the family* $(L_\varepsilon)_\varepsilon$ *only in the following sense*: *if* $(N_\varepsilon)_\varepsilon$ *is a negligible net of maps, then the extensions of* $(L_\varepsilon)_\varepsilon$ *and* $(L_\varepsilon + N_\varepsilon)_\varepsilon$ *are equal.*

(iii) *Let* (a, b, c) *be in* $(\mathbb{R}_+)^3$: *if the net* $(L_\varepsilon)_\varepsilon$ *is* $\mathcal{L}_{b,c}$*-strongly continuously moderate, then* $L(\mathcal{G}_{\mathcal{L}_a,C}(\mathbb{R}^n))$ *is included in* $\mathcal{G}_{\mathcal{L}_{ab+c}}(\mathbb{R}^m)$. *Moreover,* $L(\mathcal{G}_{\mathcal{L}_0,C}(\mathbb{R}^n))$ *is included in* $\mathcal{G}_{\mathcal{L}_c}(\mathbb{R}^m)$ *even if* $b = +\infty$.

Proof. (i) Fix $K \Subset \mathbb{R}^m$, $l \in \mathbb{N}$ and let $(f_\varepsilon)_\varepsilon$ be in $\mathcal{X}(\mathcal{D}(\mathbb{R}^n))$. There exists $j \in \mathbb{N}$ such that $(f_\varepsilon)_\varepsilon \in \mathcal{X}_j(\mathbb{R}^n)$ and, according to the definition of moderate nets, we get $(C_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{R}_+)$ and $l' \in \mathbb{N}$ such that
$$p_{K,l}\big(L_\varepsilon(f_\varepsilon)\big) \leqslant C_\varepsilon p_{j,l'}(f_\varepsilon) \quad \text{for } \varepsilon \text{ small enough.} \tag{10}$$
Inequality (10) leads to $(L_\varepsilon(f_\varepsilon))_\varepsilon \in \mathcal{X}(\mathrm{C}^\infty(\mathbb{R}^m))$. Moreover, if $(f_\varepsilon)_\varepsilon$ belongs to $\mathcal{N}(\mathcal{D}(\mathbb{R}^n))$, the same inequality implies that $(L_\varepsilon(f_\varepsilon))_\varepsilon \in \mathcal{N}(\mathrm{C}^\infty(\mathbb{R}^m))$. These two properties show that L is well defined by formula (9).

(ii) The proof is straightforward, using arguments similar to those used for the first assertion.

(iii) We shall do the proof for $a \in (0, +\infty)$. Suppose that $(L_\varepsilon)_\varepsilon$ is $\mathcal{L}_{b,c}$-strongly moderate and consider $(f_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{L}_a}(\mathrm{C}^\infty(\mathbb{R}^n)) \cap \mathcal{X}_j(\mathbb{R}^n)$. Fix $K \Subset \mathbb{R}^m$. There exists a sequence $\lambda \in \mathbb{N}^\mathbb{N}$, with $\limsup_{l\to+\infty}(\lambda(l)/l) < b$, and a sequence $r \in \mathbb{N}^\mathbb{N}$, with $\limsup_{l\to+\infty}(r(l)/l) < c$, such that
$$\forall l \in \mathbb{N},\ \exists C \in \mathbb{R}_+, \quad p_{K,l}\big(L_\varepsilon(f_\varepsilon)\big) \leqslant C\varepsilon^{-r(l)} p_{j,\lambda(l)}(f_\varepsilon) \quad (\text{for } \varepsilon \text{ small enough}).$$
As $(f_\varepsilon)_\varepsilon$ is in $\mathcal{X}_{\mathcal{L}_a}(\mathrm{C}^\infty(\mathbb{R}^n))$, there exists a sequence $q \in \mathbb{N}^\mathbb{N}$, with $\limsup_{\lambda\to+\infty}(q(\lambda)/\lambda) < a$, such that
$$\forall \lambda \in \mathbb{N}, \quad p_{j,\lambda}(f_\varepsilon) = \mathrm{O}\big(\varepsilon^{-q(\lambda)}\big) \quad \text{for } \varepsilon \to 0.$$

We get that

$$\forall l \in \mathbb{N}, \quad p_{K,l}\big(L_\varepsilon(f_\varepsilon)\big) = \mathrm{O}\big(\varepsilon^{-q_1(l)}\big) \quad \text{for } \varepsilon \to 0, \text{ with } q_1(l) = r(l) + q\big(\lambda(l)\big).$$

If $\lambda(l)$ is bounded, we have immediately that $q_1(l)/l = \mathrm{O}(r(l)/l)$ for $l \to +\infty$. If $\lambda(l)$ is not bounded, for $\lambda(l) \neq 0$,

$$\frac{q_1(l)}{l} = \frac{r(l)}{l} + \frac{q(\lambda(l))}{\lambda(l)} \frac{\lambda(l)}{l}. \tag{11}$$

We have $\limsup_{l\to+\infty}(q(\lambda(l))/\lambda(l)) < a$ and thus $\limsup_{l\to+\infty} \frac{q(\lambda(l))}{\lambda(l)} \frac{\lambda(l)}{l} < ab$. This gives

$$\limsup_{l\to+\infty}\big(q_1(l)/l\big) < ab + c$$

and $(L_\varepsilon(f_\varepsilon))_\varepsilon \in \mathcal{X}_{\mathcal{L}_{ab+c}}(\mathrm{C}^\infty(\mathbb{R}^m))$, which shows the assertion.

Finally, if $(f_\varepsilon)_\varepsilon$ is in $\mathcal{X}_{\mathcal{L}_0}(\mathrm{C}^\infty(\mathbb{R}^n))$, the sequence q can be chosen such that $\lim_{\lambda\to+\infty}(q(\lambda)/\lambda) = 0$. Then, for $b = +\infty$, the sequence $l \mapsto \lambda(l)/l$ is bounded. It follows that $\limsup_{l\to+\infty}(q_1(l)/l) < c$. □

We can weaken the assumption on the family $(L_\varepsilon)_\varepsilon$, if we accept that the result of assertion (iii) of Proposition 21 holds in a smaller space. More precisely:

Proposition 22. *With the notations of Proposition* 21*, if the family $(L_\varepsilon)_\varepsilon$ is moderate, with the assumption that the net of constants $(C_\varepsilon)_\varepsilon$ in* (7) *satisfies $C_\varepsilon = \mathrm{O}(\varepsilon^{-r(l)})$ with $\limsup_{l\to+\infty}(r(l)/l) < c$, then the extension L satisfies $L(\mathcal{G}_C^\infty(\mathbb{R}^n)) \subset \mathcal{G}_{\mathcal{L}_c}(\mathbb{R}^m)$.*

The proof is a simplification of the one of Proposition 21(iii).

4.2. Main theorems

Theorem 23. *Consider $(a,b,c) \in (\mathbb{R}_+)^3$ such that $a \leqslant 1$ and $ab + c \leqslant 1$. Let $(L_\varepsilon)_\varepsilon \in \mathcal{L}(\mathcal{D}(\mathbb{R}^n), \mathrm{C}^\infty(\mathbb{R}^m))^{(0,1]}$ be a net of $\mathcal{L}_{b,c}$-strongly continuously moderate linear maps, and $L \in \mathcal{L}(\mathcal{G}_C(\mathbb{R}^n), \mathcal{G}(\mathbb{R}^m))$ its canonical extension. There exists $H_L \in \mathcal{G}(\mathbb{R}^m \times \mathbb{R}^n)$ such that*

$$\forall f \in \mathcal{G}_{\mathcal{L}_a,C}\big(\mathbb{R}^n\big), \quad L(f) = \left[\left(x \longmapsto \int H_{L,\varepsilon}(x,y) f_\varepsilon(y)\,dy\right)_\varepsilon\right], \tag{12}$$

where $(H_{L,\varepsilon})_\varepsilon$ (respectively $(f_\varepsilon)_\varepsilon$) is any representative of H_L (respectively f).

The parameters (a,b,c) can be interpreted in the following way: the parameters b and c give the "regularity" of the net $(L_\varepsilon)_\varepsilon$, with respect to the derivative index l in the family of seminorms $(p_{K,l})_{K,l}$ for b, and to the parameter ε for c. The more "irregular" the net of maps $(L_\varepsilon)_\varepsilon$ is (that is: the bigger b is and the closer to 1 c is), the smaller is the space on which equality (12) holds. The limit cases for c are $c = 1$ (for which $a = 0$ and (12) holds only on $\mathcal{G}_{\mathcal{L}_0,C}(\mathbb{R}^n)$) and $c = 0$ (the net of constants $(C_\varepsilon)_\varepsilon$ in relation (7) depends slowly on ε) for which the conditions on (a,b,c) are reduced to $a < 1$ and $ab \leqslant 1$. (Note that these limiting conditions are induced by Lemma 17.) By using Proposition 22, we can give a version of Theorem 23 valid for more irregular nets of maps.

Theorem 24. *Let* $(L_\varepsilon)_\varepsilon \in \mathcal{L}(\mathcal{D}(\mathbb{R}^n), \mathrm{C}^\infty(\mathbb{R}^m))^{(0,1]}$ *be a net of continuously moderate linear maps such that the net of constants* $(C_\varepsilon)_\varepsilon$ *in relation* (7) *satisfies* $C_\varepsilon = \mathrm{O}(\varepsilon^{-r(l)})$ *with*

$$\limsup_{l\to+\infty}\bigl(r(l)/l\bigr) < 1$$

and $L \in \mathcal{L}(\mathcal{G}_C(\mathbb{R}^n), \mathcal{G}(\mathbb{R}^m))$ *its canonical extension. Then, the conclusion of Theorem* 23 *holds on* $\mathcal{G}_C^\infty(\mathbb{R}^n)$.

Example 25. Remark 18 and relation (6) show also that, for $a \in [0,1]$, the identity map of $\mathcal{G}_{\mathcal{L}_a,C}(\mathbb{R}^n))$ admits as kernel

$$\Phi = \bigl[\bigl((x,y) \mapsto \varphi_\varepsilon(x-y)\bigr)_\varepsilon\bigr], \tag{13}$$

where $(\varphi_\varepsilon)_{\varepsilon\in(0,1]}$ is any net of mollifiers satisfying conditions (3) and (4) of Lemma 9.

This example shows also that, in general, we do not have uniqueness in Theorem 23, but a so-called *weak uniqueness*. In our example, any net $(\varphi_\varepsilon)_\varepsilon$ of mollifiers satisfying conditions (3) and (4) verify also $\varphi_\varepsilon \to \delta$ in $\mathcal{D}'$ for $\varepsilon \to 0$. Thus, kernels of the form (13) are associated in $\mathcal{G}(\mathbb{R}^m \times \mathbb{R}^n)$, or weakly equal, i.e., the difference of their representative tends to 0 in $\mathcal{D}'$ for $\varepsilon \to 0$. (See [7,10,11] for further analysis of different associations in Colombeau type spaces.)

4.3. Relationship with the classical Schwartz theorem: equality in generalized distribution sense

Let $\Lambda \in \mathcal{L}(\mathcal{D}(\mathbb{R}^n), \mathcal{D}'(\mathbb{R}^m))$ be continuous for the strong topology and consider the family of linear mappings $(L_\varepsilon)_\varepsilon$ defined by

$$L_\varepsilon : \mathcal{D}\bigl(\mathbb{R}^n\bigr) \to \mathrm{C}^\infty\bigl(\mathbb{R}^m\bigr), \quad f \mapsto \Lambda(f) * \varphi_{\varepsilon^s} \quad (s \text{ real parameter in } (0,1)),$$

where $(\varphi_\varepsilon)_\varepsilon$ is a family of mollifiers satisfying conditions (3) and (4) of Lemma 9. We have:

Proposition 26.

(i) *For all* $\varepsilon \in (0,1]$, L_ε *is continuous for the usual topologies of* $\mathcal{D}(\mathbb{R}^n)$ *and* $\mathrm{C}^\infty(\mathbb{R}^m)$.
(ii) *The net* $(L_\varepsilon)_\varepsilon$ *is* $(0,s)$*-strongly moderate.*

Consequently, Theorem 23 shows that the canonical extension L of the net $(L_\varepsilon)_\varepsilon$ admits a kernel H_L.

Proposition 27. *For all* $f \in \mathcal{D}(\mathbb{R}^n)$, $\Lambda(f)$ *is equal to* $\tilde{H}_L(f)$ *in the* generalized distribution sense, *that is*

$$\forall \Phi \in \mathcal{D}\bigl(\mathbb{R}^m\bigr), \quad \bigl\langle \Lambda(f), \Phi\bigr\rangle = \bigl\langle \tilde{H}_L(f), \Phi\bigr\rangle \quad \text{in } \bar{\mathbb{C}}.$$

In other words, this generalized distributional equality (introduced in [11]) means that, for all $k \in \mathbb{N}$,

$$\forall \Phi \in \mathcal{D}\big(\mathbb{R}^m\big), \quad \big\langle \Lambda(f), \Phi \big\rangle - \int \bigg(\int H_{L,\varepsilon}(x,y) f(y)\,dy \bigg) \Phi(x)\,dx = \mathrm{O}\big(\varepsilon^k\big),$$
$$\text{for } \varepsilon \to 0, \tag{14}$$

where $(H_{L,\varepsilon})_\varepsilon$ is any representative of H_L.

In particular, this result implies that $\Lambda(f)$ and $\tilde{H}_L(f)$ are associated or weakly equal, i.e.,

$$\bigg(x \mapsto \int H_{L,\varepsilon}(x,y) f(y)\,dy \bigg) \longrightarrow \Lambda(f) \quad \text{in } \mathcal{D}' \text{ for } \varepsilon \to 0.$$

5. Proofs of Theorem 23 and Propositions 26 and 27

5.1. *Proof of Theorem 23*

We shall only prove Theorem 23, since the proof of Theorem 24 follows the same lines. Let us fix a net of mollifiers $(\varphi_\varepsilon)_\varepsilon \in (\mathcal{D}(\mathbb{R}^m))^{(0,1]}$ (respectively $(\psi_\varepsilon)_\varepsilon \in (\mathcal{D}(\mathbb{R}^m))^{(0,1]}$) satisfying conditions (3) and (4) of Lemma 9. For all $y \in \mathbb{R}^n$, we define

$$\psi_{\varepsilon,\cdot} : \mathbb{R}^n \to \mathcal{D}\big(\mathbb{R}^n\big), \quad y \mapsto \psi_{\varepsilon,y} = \big\{ v \mapsto \psi_\varepsilon(y - v) \big\}.$$

For all $y \in \mathbb{R}^n$ and $\varepsilon \in (0,1]$, we set $\Psi_{\varepsilon,y} = L_\varepsilon(\psi_{\varepsilon,y})$.

Lemma 28. *The map*

$$\Psi_\varepsilon : \mathbb{R}^n \to \mathrm{C}^\infty\big(\mathbb{R}^m\big), \quad y \mapsto \Psi_{\varepsilon,y} = L_\varepsilon(\psi_{\varepsilon,y}),$$

is of class C^∞ *for all* $\varepsilon \in (0,1]$.

Proof. The map $(y,v) \mapsto \psi_\varepsilon(y-v)$ from $\mathbb{R}^{2n}$ to $\mathbb{R}$ is clearly of class C^∞. It follows that the map $\psi_{\varepsilon,\cdot} : y \mapsto \psi_{\varepsilon,y}$, considered as a map from $\mathbb{R}^n$ to $\mathrm{C}^\infty(\mathbb{R}^n)$, is C^∞. (See, for example, [7, Theorem 2.2.2].) As each $\psi_{\varepsilon,y}$ is compactly supported, we can show that $\psi_{\varepsilon,\cdot}$ belongs in fact to $\mathrm{C}^\infty(\mathbb{R}^n, \mathcal{D}(\mathbb{R}^n))$ by using local arguments. Since L_ε is linear and continuous it follows that Ψ_ε is C^∞. □

Let us define, for all $\varepsilon \in (0,1]$ and $(x,y) \in \mathbb{R}^m \times \mathbb{R}^n$:

$$H_\varepsilon(x,y) = (\Psi_{\varepsilon,y} * \varphi_\varepsilon)(x) = \int L_\varepsilon(\psi_{\varepsilon,y})(x - \lambda)\varphi_\varepsilon(\lambda)\,d\lambda.$$

Note that this integral is performed on a fixed relatively compact set containing $\operatorname{supp} \varphi_\varepsilon$ for all $\varepsilon \in (0,1]$.

Lemma 29. *For all* $\varepsilon \in (0,1]$, H_ε *is of class* C^∞ *and* $(H_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{R}^m \times \mathbb{R}^n)$.

Proof. First, the map $g \mapsto g * \varphi_\varepsilon$ from $\mathrm{C}^\infty(\mathbb{R}^m)$ into itself is linear continuous and therefore C^∞. Using Lemma 28, we get that the map $y \mapsto (\Psi_{\varepsilon,y} * \varphi_\varepsilon) = H_\varepsilon(\cdot, y)$ from $\mathbb{R}^n$ to $\mathrm{C}^\infty(\mathbb{R}^m)$ is C^∞. Using again [7, Theorem 2.2.2], we get that H_ε belongs to $\mathrm{C}^\infty(\mathbb{R}^{m+n})$.

Consider K and K' two compact subsets of $\mathbb{R}^n$. Let us recall that the support of ψ_ε is compact and decreasing to $\{0\}$, when ε tends to 0. Then, there exists a compact set $K_\psi \subset \mathbb{R}^m$ such that, for all $\varepsilon \in (0,1]$, $\operatorname{supp}\psi_\varepsilon \subset K_\psi$ and $\operatorname{supp}\psi_{\varepsilon,y} \subset y - K_\psi$. Moreover, we can find a compact set K_j (notations are those of 4.1) such that

$$\forall \varepsilon \in (0,1],\ \forall y \in K', \quad \psi_{\varepsilon,y} \in \mathcal{D}_j\big(\mathbb{R}^n\big),$$

and $p_{j,l}(\psi_{\varepsilon,y}) = p_{K_\psi,l}(\psi_\varepsilon)$ for all $\varepsilon \in (0,1]$.

Let us now consider $(\alpha,\beta) \in (\mathbb{N}^n)^2$ and ∂^α (respectively ∂^β) the α-partial derivative (respectively β-partial derivative) with respect to the variable x (respectively y). Noticing that there exists a compact set $K_\varphi \subset \mathbb{R}^m$ such that, for all $\varepsilon \in (0,1]$, $\operatorname{supp}\varphi_{\varepsilon,y} \subset K_\varphi$, we get the existence of a constant C such that, for all $\varepsilon \in (0,1]$,

$$\begin{aligned}\forall (x,y) \in K \times K', \quad \big|\partial^\alpha \partial^\beta H_\varepsilon(x,y)\big| &\leqslant C \sup_{\xi \in x - K_\varphi} \big|\partial^\beta L_\varepsilon(\psi_{\varepsilon,y})(\xi)\big| \sup_{\xi \in K_\varphi} \big|\partial^\alpha \varphi_\varepsilon(\xi)\big|, \\ &\leqslant C p_{K-K_\varphi,|\beta|}\big(L_\varepsilon(\psi_{\varepsilon,y})\big) p_{K_\varphi,|\alpha|}(\varphi_\varepsilon).\end{aligned}$$

The moderateness of $(L_\varepsilon)_\varepsilon$ implies the existence of $l \in \mathbb{N}$ and $(C'_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{R}_+)$ such that, for all $\varepsilon \in (0,1]$,

$$\begin{aligned}\forall (x,y) \in K \times K', \quad \big|\partial^\alpha \partial^\beta H_\varepsilon(x,y)\big| &\leqslant C'_\varepsilon\, p_{j,l}(\psi_{\varepsilon,y}) p_{K\varphi,|\alpha|}(\varphi_\varepsilon) \\ &\leqslant C'_\varepsilon p_{K_\psi,l}(\psi_\varepsilon) p_{K_\varphi,|\alpha|}(\varphi_\varepsilon).\end{aligned}$$

The last inequality shows that $(p_{K\times K'|\alpha|+|\beta|}(H_\varepsilon))_\varepsilon$ belongs to $\mathcal{X}(\mathbb{R}_+)$, which concludes the proof. □

For all $(f_\varepsilon)_\varepsilon$ in $\mathcal{X}(\mathcal{D}(\mathbb{R}^n))$ (defined in (1)) we can consider

$$\tilde{H}_\varepsilon(f_\varepsilon)(x) = \int H_\varepsilon(x,y) f_\varepsilon(y)\, dy = \int \left(\int L_\varepsilon(\psi_{\varepsilon,y})(x-\lambda)\varphi_\varepsilon(\lambda)\, d\lambda \right) f_\varepsilon(y)\, dy,$$

since for all $\varepsilon \in (0,1]$, f_ε is compactly supported.

Lemma 30. *For all $(f_\varepsilon)_\varepsilon$ in $\mathcal{X}(\mathcal{D}(\mathbb{R}^n))$, we have*

$$\tilde{H}_\varepsilon(f_\varepsilon)(x) = \big(L_\varepsilon(\psi_\varepsilon * f_\varepsilon) * \varphi_\varepsilon\big)(x).$$

Proof. Let $(f_\varepsilon)_\varepsilon$ be in $\mathcal{X}(\mathcal{D}(\mathbb{R}^n))$. For all $\varepsilon \in (0,1]$ and $x \in \mathbb{R}^m$, we have

$$\begin{aligned}\tilde{H}_\varepsilon(f_\varepsilon)(x) &= \int_{\operatorname{supp} f} \left(\int_{\operatorname{supp}\varphi_\varepsilon} L_\varepsilon(\psi_{\varepsilon,y})(x-\lambda)\varphi_\varepsilon(\lambda)\, d\lambda \right) f_\varepsilon(y)\, dy \\ &= \int_{\operatorname{supp}\varphi_\varepsilon} \int_{\operatorname{supp} f} L_\varepsilon(\psi_{\varepsilon,y})(x-\lambda)\varphi_\varepsilon(\lambda) f_\varepsilon(y)\, d\lambda\, dy \\ &= \int \left(\int L_\varepsilon(\psi_{\varepsilon,y})(x-\lambda) f_\varepsilon(y)\, dy \right) \varphi_\varepsilon(\lambda)\, d\lambda,\end{aligned}$$

the two last equalities being true by Fubini's theorem, each integral being calculated on a compact set.

For all $\varepsilon \in (0,1]$ and $\xi \in \mathbb{R}^m$, we have the following equality:

$$\begin{aligned}\int L_\varepsilon(\psi_{\varepsilon,y})(\xi) f_\varepsilon(y)\,dy &= L_\varepsilon\left(v \mapsto \int \psi_{\varepsilon,y}(v) f_\varepsilon(y)dy\right)(\xi)\\ &= L_\varepsilon\left(v \mapsto \int \psi_\varepsilon(y-v) f_\varepsilon(y)dy\right)(\xi).\end{aligned}$$

Indeed, the integrals under consideration in the above equalities are integrals of continuous functions on compact sets and can be considered as limits of Riemann sums in the spirit of [8, Lemma 4.1.3, p. 89]:

$$\forall \xi \in \mathbb{R}^m, \quad \int L_\varepsilon(\psi_{\varepsilon,y})(\xi) f_\varepsilon(y)\,dy = \lim_{h\to 0} \sum_{k\in\mathbb{Z}} h^n L_\varepsilon\big(\psi_\varepsilon(kh-v)\big)(\xi) f_\varepsilon(kh)$$

$$\forall v \in \mathbb{R}^n, \quad \int \psi_\varepsilon(y-v) f_\varepsilon(y)dy = \lim_{h\to 0} \sum_{k\in\mathbb{Z}} h^n \psi_\varepsilon(kh-v) f_\varepsilon(kh).$$

Notice that, in both sums over $\mathbb{Z}$, only a finite number of terms are nonzero, since each f_ε is compactly supported. Thus, as the mapping L_ε is linear, we have

$$L_\varepsilon\left(\sum_{k\in\mathbb{Z}} \psi_\varepsilon(kh-v) f_\varepsilon(kh)\right) = \sum_{k\in\mathbb{Z}} f_\varepsilon(kh) L_\varepsilon\big(\psi_\varepsilon(kh-v)\big).$$

By continuity of L_ε, we get

$$\begin{aligned}L_\varepsilon\left(\int \psi_\varepsilon(y-v) f_\varepsilon(y)dy\right)(\xi) &= L_\varepsilon\left(\lim_{h\to 0} \sum_{k\in\mathbb{Z}} h^n \psi_\varepsilon(kh-v) f_\varepsilon(kh)\right)(\xi)\\ &= \lim_{h\to 0}\left(\sum_{k\in\mathbb{Z}} f_\varepsilon(kh) L_\varepsilon\big(\psi_\varepsilon(kh-v)\big)(\xi)\right)\\ &= \int L_\varepsilon(\psi_{\varepsilon,y})(\xi) f_\varepsilon(y)\,dy.\end{aligned}$$

Finally, we get, for all $\varepsilon \in (0,1]$ and $\xi \in \mathbb{R}^m$,

$$\int L_\varepsilon(\psi_{\varepsilon,y})(\xi) f_\varepsilon(y)\,dy = L_\varepsilon\left(\int \psi_\varepsilon(y-v) f_\varepsilon(y)dy\right)(\xi) = L_\varepsilon(\psi_\varepsilon * f_\varepsilon)(\xi),$$

and

$$\tilde{H}_\varepsilon(f_\varepsilon)(x) = \int L_\varepsilon(\psi_\varepsilon * f_\varepsilon)(x-\lambda)\varphi_\varepsilon(\lambda)\,d\lambda = \big(L_\varepsilon(\psi_\varepsilon * f_\varepsilon) * \varphi_\varepsilon\big)(x). \qquad \square \tag{15}$$

We now complete the proof of Theorem 23. Set

$$H_L = (H_\varepsilon)_\varepsilon + \mathcal{N}\big(C^\infty\big(\mathbb{R}^{m+n}\big)\big) = \big((x,y) \mapsto (\Psi_{\varepsilon,y} * \varphi_\varepsilon)(x)\big)_\varepsilon + \mathcal{N}\big(C^\infty\big(\mathbb{R}^{m+n}\big)\big).$$

For all $(f_\varepsilon)_\varepsilon$ in $\mathcal{X}_{\mathcal{L}_0}(\mathcal{D}(\mathbb{R}^n))$, we have

$$\tilde{H}_L\big(\big[(f_\varepsilon)_\varepsilon\big]\big) = \big[\big(\tilde{H}_\varepsilon(f_\varepsilon)\big)_\varepsilon\big],$$

by definition of the integral in $\mathcal{G}(\mathbb{R}^n)$. We have to compare $(\tilde{H}_\varepsilon(f_\varepsilon))_\varepsilon$ and $(L_\varepsilon(f_\varepsilon))_\varepsilon$. According to Lemma 30, we have for all $\varepsilon \in (0,1]$,

$$\begin{aligned}\tilde{H}_\varepsilon(f_\varepsilon) - L_\varepsilon(f_\varepsilon) &= \big(L_\varepsilon(\psi_\varepsilon * f_\varepsilon) * \varphi_\varepsilon\big) - L_\varepsilon(f_\varepsilon)\\ &= L_\varepsilon(\psi_\varepsilon * f_\varepsilon) * \varphi_\varepsilon - L_\varepsilon(f_\varepsilon) * \varphi_\varepsilon + L_\varepsilon(f_\varepsilon) * \varphi_\varepsilon - L_\varepsilon(f_\varepsilon)\\ &= L_\varepsilon(\psi_\varepsilon * f_\varepsilon - f_\varepsilon) * \varphi_\varepsilon + L_\varepsilon(f_\varepsilon) * \varphi_\varepsilon - L_\varepsilon(f_\varepsilon).\end{aligned}$$

Remarking that $(f_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{L}_a}(\mathrm{C}^\infty(\Omega))$ and $(L_\varepsilon(f_\varepsilon))_\varepsilon \in \mathcal{X}_{\mathcal{L}_{a+bc}}(\mathrm{C}^\infty(\Omega)) \subset \mathcal{X}_{\mathcal{L}_1}(\mathrm{C}^\infty(\Omega))$, we get $(L_\varepsilon(f_\varepsilon) * \varphi_\varepsilon - L_\varepsilon(f_\varepsilon))_\varepsilon \in \mathcal{N}(\mathrm{C}^\infty(\mathbb{R}^m))$ and $(\psi_\varepsilon * f_\varepsilon - f_\varepsilon)_\varepsilon \in \mathcal{N}(\mathrm{C}^\infty(\mathbb{R}^m))$ by Lemma 17. This last property gives

$$\begin{aligned}&\big(L_\varepsilon(\psi_\varepsilon * f_\varepsilon - f_\varepsilon)\big)_\varepsilon \in \mathcal{N}\big(\mathrm{C}^\infty\big(\mathbb{R}^m\big)\big) \quad \text{and}\\ &\big(L_\varepsilon(\psi_\varepsilon * f_\varepsilon - f_\varepsilon) * \varphi_\varepsilon\big) \in \mathcal{N}\big(\mathrm{C}^\infty\big(\mathbb{R}^m\big)\big),\end{aligned}$$

since $(\eta_\varepsilon * \varphi_\varepsilon)_\varepsilon \in \mathcal{N}(\mathrm{C}^\infty(\mathbb{R}^m))$ for all $(\eta_\varepsilon)_\varepsilon \in \mathcal{N}(\mathrm{C}^\infty(\mathbb{R}^m))$. Finally

$$\big[\big(\tilde{H}_\varepsilon(f_\varepsilon)\big)_\varepsilon\big] = \big[\big(L_\varepsilon(f_\varepsilon)\big)_\varepsilon\big] = L\big(\big[(f_\varepsilon)_\varepsilon\big]\big),$$

this last equality by definition of the extension of a linear map.

5.2. *Proof of Proposition 26*

(i) We have only to prove continuity on 0. Let us fix $\varepsilon \in (0,1]$. Take $(f_k)_k \in \mathcal{D}(\mathbb{R}^n)^{\mathbb{N}}$ a sequence converging to 0 in $\mathcal{D}(\mathbb{R}^n)$. Since Λ is continuous, the sequence $(T_k)_k = (\Lambda(f_k))_k$ tends to 0 in $\mathcal{D}'(\mathbb{R}^m)$ for the strong topology. Let us recall that [16]:

Lemma 31. *A sequence $(T_k)_k$ tends to* 0 *in $\mathcal{D}'(\mathbb{R}^m)$ for the strong topology if and only if for all $\theta \in \mathcal{D}(\mathbb{R}^m)$ the sequence $(T_k * \theta)_k$ tends to* 0*, uniformly on every compact set.*

For all α in $\mathbb{N}^m$, we take $\theta_\alpha = \partial^\alpha \varphi_{\varepsilon^s}$. Applying Lemma 31, the sequences

$$\big(T_k * \partial^\alpha \varphi_{\varepsilon^s}\big)_k = \big(\partial^\alpha (T_k * \varphi_{\varepsilon^s})\big)_k$$

tend to 0, uniformly on each compact set of $\mathbb{R}^m$. Thus, L_ε is continuous.

(ii) According to Definition 20, we have to show that, for all $j \in \mathbb{N}$, the net $(L_{\varepsilon|\mathcal{D}_j})_\varepsilon \in (\mathcal{L}(\mathcal{D}_j(\mathbb{R}^n), \mathrm{C}^\infty(\mathbb{R}^m)))^{(0,1]}$ is strongly moderate. We have

$$\begin{aligned}&\forall f \in \mathcal{D}_j\big(\mathbb{R}^n\big),\ \forall x \in \mathbb{R}^m,\ \forall \alpha \in \mathbb{N}^m,\\ &\quad \partial^\alpha\big(L_{\varepsilon|\mathcal{D}_j}(f)\big)(x) = \big(\Lambda(f) * \partial^\alpha \varphi_{\varepsilon^s}\big)(x) = \big\langle \Lambda(f), \big\{y \mapsto \partial^\alpha \varphi_{\varepsilon^s}(x-y)\big\}\big\rangle.\end{aligned}$$

Consider K a compact subset of $\mathbb{R}^m$. As $\operatorname{supp} \varphi_{\varepsilon^s}$ decreases to $\{0\}$ for $\varepsilon \to 0$, there exists a compact set K' such that

$$\forall x \in K,\ \forall \varepsilon \in (0,1], \quad \operatorname{supp}\big(\partial^\alpha\big(y \mapsto \varphi_{\varepsilon^s}(x-y)\big)\big) \subset K'.$$

The map

$$\Theta : \mathcal{D}_j\big(\mathbb{R}^n\big) \times \mathcal{D}_{K'}\big(\mathbb{R}^m\big), \quad (f,\varphi) \to \big\langle \Lambda(f), \varphi(x - \cdot)\big\rangle$$

is a bilinear map, separately continuous since Λ is continuous. As $\mathcal{D}_j(\mathbb{R}^n)$ and $\mathcal{D}_{K'}(\mathbb{R}^m)$ are Fréchet spaces, Θ is globally continuous. There exists $C > 0$, $l_1 \in \mathbb{N}$, $l_2 \in \mathbb{N}$ such that

$$\forall (f, \varphi) \in \mathcal{D}_j(\mathbb{R}^n) \times \mathcal{D}_{K'}(\mathbb{R}^m), \quad \left|\left\langle \Lambda(f), \varphi \right\rangle\right| \leqslant C P_{j,l_1}(f) P_{K',l_2}\left(\varphi(x - \cdot)\right).$$

In particular, for $l \in \mathbb{N}$ and $\alpha \in \mathbb{N}^m$ with $|\alpha| \leqslant l$, we have

$$\left|\left\langle \Lambda(f), \partial^\alpha \varphi_\varepsilon(x - \cdot)\right\rangle\right| \leqslant C P_{j,l_1}(f) P_{K',l_2}\left(\partial^\alpha \varphi_{\varepsilon^s}(x - \cdot)\right), \tag{16}$$

and $P_{K',l_2}(\partial^\alpha \varphi_{\varepsilon^s}(x - \cdot)) \leqslant P_{K',l_2+l}(\partial^\alpha \varphi_{\varepsilon^s}(x - \cdot))$.

Let us recall that

$$\partial^\alpha \varphi_{\varepsilon^s}(x - \cdot) = \partial^\alpha \left\{ y \mapsto \varepsilon^{-sm} \varphi\left((x - y)/\varepsilon^s\right) \kappa\left(|\ln \varepsilon|(x - y)\right)\right\}.$$

By induction on $|\alpha|$ and using the boundedness of φ, κ and their derivatives on $\mathbb{R}^m$, we can show that there exists a constant C_1, depending on $|\alpha|$, φ and κ and their derivatives but not on ε, such that

$$\sup_{y \in K'} \left|\partial^\alpha \left\{ y \mapsto \varphi_{\varepsilon^s}(x - y)\right\}\right| \leqslant C_1' \, \varepsilon^{-s(m+|\alpha|+1)}.$$

It follows that there exists a constant C_2 (independent of ε) such that

$$P_{K',l_2+l}\left(\varphi_\varepsilon(x - \cdot)\right) \leqslant C_2 \, \varepsilon^{-s(m+l_2+l+1)}.$$

Inserting this result into Eq. (16), we finally get the existence of a constant C_3 (independent of ε) such that

$$p_{K,l}\left(L_{\varepsilon|\mathcal{D}_j}(f)\right) = \sup_{x \in K,\, |\alpha| \leqslant l} \left|\left\langle \Lambda(f), \partial^\alpha \varphi_\varepsilon(x - \cdot)\right\rangle\right| \leqslant C_3 \, \varepsilon^{-s(m+l_2+l+1)} P_{j,l_1}(f).$$

The sequence $r(\cdot) = \{l \mapsto s(m + l_2 + l + 1)\}$ satisfies $\lim_{l \to +\infty}(r(l)/l) = s < 1$. Recalling that l_1 does not depend on l, we obtain our claim. $\square$

5.3. Proof of Proposition 27

We first have the following:

Lemma 32. *For all $T \in \mathcal{D}'(\mathbb{R}^m)$, $[(T * \varphi_{\varepsilon^s})_\varepsilon]$ is equal to T in the generalized distribution sense.*

Proof. Take $T \in \mathcal{D}'(\mathbb{R}^m)$ and $g \in \mathcal{D}(\mathbb{R}^m)$. Set, for $\varepsilon \in (0, 1]$ and for K such that $\operatorname{supp} g \subset K \Subset \mathbb{R}^m$,

$$A_{\varepsilon^s} = \int_K (T * \varphi_{\varepsilon^s})(x) g(x)\, dx = \int_K \left\langle T, \varphi_{\varepsilon^s}(x - \cdot)\right\rangle g(x)\, dx.$$

As $\operatorname{supp} \varphi_{\varepsilon^s}$ decrease to $\{0\}$ for $\varepsilon \to 0$, there exists a relatively compact open subset Ω such that

$$\forall x \in K,\ \forall \varepsilon \in (0, 1], \quad \operatorname{supp}\left(y \mapsto \varphi_{\varepsilon^s}(x - y)\right) \subset \Omega.$$

There exists f continuous with compact support and $\alpha \in \mathbb{N}^m$ such that $T_{|\Omega} = \partial^\alpha f$. This implies that $\langle T, \varphi_{\varepsilon^s}(x - \cdot)\rangle = \langle \partial^\alpha f, \varphi_{\varepsilon^s}(x - \cdot)\rangle$ and

$$(T * \varphi_{\varepsilon^s})(x) = \left(\partial^\alpha f * \varphi_{\varepsilon^s}\right)(x) = \partial^\alpha (f * \varphi_{\varepsilon^s})(x).$$

By integration by part (g is compactly supported), it follows that

$$A_{\varepsilon^s} = \int_K \partial^\alpha (f * \varphi_{\varepsilon^s})(x) g(x)\, dx = (-1)^{|\alpha|} \int_K (f * \varphi_{\varepsilon^s})(x) \partial^\alpha g(x)\, dx.$$

Consider now an integer k and $\beta \in \mathbb{N}^m$ such that $\beta = \beta_1 + \cdots + \beta_m$ with $\beta_j \geqslant k$, for each $j \in \{1, \ldots, m\}$. We consider a function F_β such that $\partial^\beta F_\beta = f$, which exists since f is continuous. This function is at least of class C^k. We have

$$\begin{aligned} A_{\varepsilon^s} &= (-1)^{|\alpha|} \int_K \left(\partial^\beta F_\beta * \varphi_{\varepsilon^s}\right)(x) \partial^\alpha g(x)\, dx \\ &= (-1)^{|\alpha|+|\beta|} \int_K (F_\beta * \varphi_{\varepsilon^s})(x) \partial^{\alpha+\beta} g(x)\, dx; \end{aligned}$$

$$\begin{aligned} \langle T, g\rangle = \langle \partial^\alpha f, g\rangle = \left\langle \partial^{\alpha+\beta} F_\beta, g\right\rangle &= (-1)^{|\alpha|+|\beta|} \left\langle F_\beta, \partial^{\alpha+\beta} g\right\rangle \\ &= (-1)^{|\alpha|+|\beta|} \int_K (F_\beta)(x) \partial^{\alpha+\beta} g(x)\, dx. \end{aligned}$$

Then

$$\langle T * \varphi_{\varepsilon^s}, g\rangle - \langle T, g\rangle = (-1)^{|\alpha|+|\beta|} \int_K \left((F_\beta * \varphi_{\varepsilon^s})(x) - (F_\beta)(x)\right) \partial^{\alpha+\beta} g(x)\, dx.$$

An adaptation (and simplification) of the proof of Lemma 17 shows that

$$(F_\beta * \varphi_{\varepsilon^s})(x) - (F_\beta)(x) = \mathrm{O}\left(\varepsilon^{ks}\right) \quad \text{for } \varepsilon \to 0.$$

As g is compactly supported, this last relation leads to

$$\langle T * \varphi_{\varepsilon^s}, g\rangle - \langle T, g\rangle = \mathrm{O}\left(\varepsilon^{ks}\right) \quad \text{for } \varepsilon \to 0.$$

Since k is arbitrary, our claim follows. □

This lemma implies that for all $f \in \mathcal{D}(\mathbb{R}^n)$, $[(L_\varepsilon(f))_\varepsilon] = [(\Lambda(f) * \varphi_{\varepsilon^s})_\varepsilon]$ is equal to $\Lambda(f)$ in the generalized distribution sense. On the other hand, according to Theorem 23, $[(L_\varepsilon(f))_\varepsilon] = \tilde{H}_L(f)$ where $\tilde{H}_L$ is the integral operator associated to the canonical extension of $(L_\varepsilon)_\varepsilon$. This ends the proof of Proposition 27. □

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